Problem 2.7

There are certain simple one-dimensional problems where the equation of motion (Newton's second law) can always be solved, or at least reduced to the problem of doing an integral. One of these (which we have met a couple of times in this chapter) is the motion of a one-dimensional particle subject to a force that depends only on the velocity v, that is, $F = F(v)$. Write down Newton's second law and separate the variables by rewriting it as $m\,dv/F(v) = dt$. Now integrate both sides of this equation and show that

$$t = m\int_{v_o}^{v} \frac{dv'}{F(v')}.$$

Provided you can do the integral, this gives t as a function of v. You can then solve to give v as a function of t. Use this method to solve the special case that $F(v) = F_o$, a constant, and comment on your result. This method of *separation of variables* is used again in Problems 2.8 and 2.9.

Solution

For a particle moving in one dimension, Newton's second law gives the equation of motion.

$$F = ma$$

Suppose the force is a function of velocity.

$$F(v) = m\frac{dv}{dt}$$

Separate variables.

$$\frac{dt}{m} = \frac{dv}{F(v)}$$

Integrate both sides definitely, assuming that at $t = 0$ the particle has velocity v_o and at some time of interest t the particle has velocity v. Because the integration is definite, no constant of integration is needed.

$$\int_0^t \frac{dt'}{m} = \int_{v_o}^{v} \frac{dv'}{F(v')}$$

Evaluate the integral on the left.

$$\frac{t}{m} = \int_{v_o}^{v} \frac{dv'}{F(v')}$$

Multiply both sides by m to get the desired result.

$$t = m\int_{v_o}^{v} \frac{dv'}{F(v')}$$

In the special case that $F(v) = F_\text{o}$, the integral on the right side can be evaluated.

$$t = m\int_{v_\text{o}}^{v} \frac{dv'}{F_\text{o}}$$

$$= \frac{m}{F_\text{o}}\int_{v_\text{o}}^{v} dv'$$

$$= \frac{m}{F_\text{o}}(v - v_\text{o}) \quad \Rightarrow \quad \begin{cases} F_\text{o}t = mv - mv_\text{o} & \text{(Impulse-momentum theorem)} \\ t = \frac{1}{a}(v - v_\text{o}) & \text{(Kinematic equation)} \end{cases}$$